



# HARMONIC ANALYSIS OF OSCILLATORS THROUGH STANDARD NUMERICAL CONTINUATION TOOLS

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In this paper, we describe a numerical continuation method that enables harmonic analysis of nonlinear periodic oscillators. This method is formulated as a boundary value problem that can be readily implemented by resorting to a standard continuation package — without modification — such as AUTO, which we used. Our technique works for any kind of oscillator, including electronic, mechanical and biochemical systems. We provide two case studies. The first study concerns itself with the autonomous electronic oscillator known as the Colpitts oscillator, and the second one with a nonlinear damped oscillator, a nonautonomous mechanical oscillator. As shown in the case studies, the proposed technique can aid both the analysis and the design of the oscillators, by following curves for which a certain constraint, related to harmonic analysis, is fulfilled.

*Keywords:* Continuation methods; harmonic analysis; nonlinear oscillator; boundary value problem.

## 1. Introduction

Continuation methods provide an efficient tool for analyzing systems of nonlinear algebraic equations whose solutions form a one-dimensional continuum. When dealing with periodic solutions of systems of ordinary differential equations (ODEs), we continue with solutions by solving a boundary value problem (BVP). We can either solve this BVP explicitly, when possible, or implicitly, by using numerical continuation tools.

The BVPs that are used to continue “standard” objects, such as equilibrium points, periodic, homoclinic and heteroclinic orbits and their bifurcations

in dynamical systems are well-known (see, for instance, [Kuznetsov, 2004] for an overview). However, it is often advantageous to formulate new BVPs to continue “nonstandard” objects, such as invariant manifolds [Doedel *et al.*, 2006], slow manifolds [Desroches *et al.*, 2008], and coherent structures such as spiral waves and other defects in oscillatory media [Bordyugov & Engel, 2007; Champneys & Sandstede, 2007].

In a recent paper [Cochelin & Vergez, 2009], a combination of harmonic analysis and continuation techniques (based on the harmonic balance method) was proposed. The main result is a new numerical continuation tool that provides standard results of

preliminary continuation analysis after a reformulation of the problem in terms of harmonic balance.

In this paper, we show how to define a novel BVP that enables harmonic analysis by using standard numerical continuation tools. The proposed BVP allows, in the spirit of [Doedel *et al.*, 2006; Desroches *et al.*, 2008; Bordyugov & Engel, 2007; Champneys & Sandstede, 2007], for “nonstandard” continuations focused on selected harmonic components of a solution without an *ad hoc* simulator like the one described in [Cochelin & Vergez, 2009].

The first application in this paper deals with the *design* of an autonomous electronic circuit (the Colpitts oscillator). Its aim is to obtain parameter charts that can be immediately understood by designers of electronic oscillators. In particular, these charts show how the limit cycle corresponding to a periodic regime changes in a familiar circuit parameter plane. The guidelines for the designer are provided, for instance, by curves corresponding to solutions with fixed amplitudes of some harmonic components or with a fixed ratio of two harmonic components.

The second application entails the harmonic *analysis* of a nonautonomous mechanical oscillator (nonlinear damped oscillator). In particular, we use the proposed BVP to analyze the so-called jump phenomenon [Schmidt & Tondl, 1986].

The main advantages of the proposed approach are the following:

- the BVP is formulated in such a way that the public-domain software package AUTO-07P [Doedel & Oldeman, 2009] can solve it;
- compared to simulations, computation times are generally lower, since numerical continuation packages operate directly on system invariant sets;
- the proposed procedure is reasonably easy to use by those who are familiar with the analysis of nonlinear dynamical systems.

On the other hand, one needs to take care of the following aspects:

- for complex systems, such as a realistic radio-frequency electronic oscillator, the procedure is effective only if preceded by a modeling phase where one defines a suitable model of the oscillator [Bizzarri *et al.*, 2010] — such a model should be as simple as possible, but able to capture the essential features of the system;

- those who are not familiar with numerical continuation packages require a preliminary training.

This paper is organized as follows. Section 2 summarizes the basic elements that the proposed technique is based on. The BVP formulation is described in Sec. 3, whereas Secs. 4 and 5 are devoted to two case studies. Some conclusions are drawn in Sec. 6.

## 2. Basic Elements

Let the following system of ODEs describe an oscillator, which can be either autonomous or nonautonomous with periodic forcing:

$$\dot{x} = \frac{dx}{dt} = g(x, t; p), \quad x, g \in \mathbb{R}^n, \quad p \in \mathbb{R}^q, \quad t \in \mathbb{R}. \quad (1)$$

When this oscillator reaches a periodic regime, it produces signals of generic period  $T$ , that is, frequency  $f = 1/T$  and angular frequency  $\omega = 2\pi f$ . The Fourier series expansion of each state variable is

$$x_j(t) = a_{0j} + \sum_{k=1}^{\infty} [a_{kj} \sin(k\omega t) + b_{kj} \cos(k\omega t)], \quad (2)$$

where the index  $j \in \{1, \dots, n\}$  selects the state variable,  $a_{0j}$  is the mean value of  $x_j(t)$  over  $T$  and we obtain the other coefficients by projecting  $x_j(t)$  on the corresponding basis functions

$$\begin{aligned} a_{kj} &= \frac{2}{T} \int_t^{t+T} x_j(\tau) \sin(k\omega\tau) d\tau \\ b_{kj} &= \frac{2}{T} \int_t^{t+T} x_j(\tau) \cos(k\omega\tau) d\tau \end{aligned} \quad (3)$$

In general, continuation methods show how system invariants (for example, equilibrium points or limit cycles) depend on one or more control parameters. One of the key elements of continuation theory is that invariant sets and their bifurcations are revealed when so-called *test functions* equal zero. Many test functions are included in the most diffused numerical continuation packages [Doedel & Oldeman, 2009; Dhooge *et al.*, 2003], but, of course, user-defined test functions can be added. For instance, in this paper we define test functions of the kind  $f(S_a, S_b, T, K_{\text{REF}})$ , where  $S_a$  and  $S_b$  denote a subset of  $n_a$  and  $n_b$  Fourier coefficients  $\{a_{kj}\}$  and  $\{b_{kj}\}$ , respectively, and  $K_{\text{REF}}$  is a constant reference value.

We can use this formulation to solve different kinds of problems. Firstly we can continue the amplitude of a harmonic component with respect to a single parameter. However, of further interest are continuations with respect to two parameters, adding a further constraint. For instance, iso-harmonic, iso-ratio, and iso-energy continuations are possible. For iso-harmonic continuations we fix the amplitude of a harmonic component, for iso-ratio continuations the ratio between amplitudes of different harmonics, and for iso-energy continuations a sum of squares of a limited number of harmonic amplitudes representing almost the whole power spectrum of the analyzed signal.

In the next section, we set up the BVP that is solved for these continuations.

### 3. Setup

We start by deriving a simplified model of the oscillator given by a small system of ODEs, as is usual when dealing with both the analysis and synthesis of a dynamical system. We call this system the *original system*. By assuming some reasonable modeling hypotheses we obtain these equations, which are expressed in terms of state variables. It is advantageous, but not compulsory, to normalize these equations and shift the origin of the normalized state space to a “significant” equilibrium point. Such an equilibrium is stable for some parameter configuration, and, by varying one parameter, undergoes a supercritical Andronov–Hopf bifurcation, which marks the appearance of a family of asymptotically stable periodic solutions evolving around the (unstable) equilibrium.

#### 3.1. The original system

For autonomous oscillators, the original system is simply the ODE system modeling the oscillator, given by  $\dot{x} = g(x(t); p)$ . For a nonautonomous oscillator with periodic forcing, we can obtain an equivalent autonomous oscillator by adding a nonlinear oscillator with the desired periodic forcing as one of the solution components (see, for instance, the AUTO-07P demo `frc` [Doedel & Oldeman, 2009; Alexander *et al.*, 1990]). In particular, for a sinusoidal forcing we can use the following secondary oscillator (very close to the normal form of the supercritical Andronov–Hopf bifurcation):

$$\begin{aligned} \dot{v} &= \alpha v + \beta w + v(v^2 + w^2) \\ \dot{w} &= -\beta v + \alpha w + w(v^2 + w^2), \end{aligned} \tag{4}$$

which for  $\alpha < 0$  asymptotically converges to the origin and for  $\alpha > 0$  has the asymptotically stable solution  $v = \sin(\beta t)$ ,  $w = \cos(\beta t)$ . For instance, if the first state variable of the original system obeys the differential equation  $\dot{x}_1 = g_1(x(t); p) + c \cos(\omega t)$  and the state of the original system is  $x = [x_1, \dots, x_n]$ , then in the corresponding autonomous system the equation for the first state variable is given by  $\dot{x}_1 = g_1(x(t); p) + cw$ , where  $\beta = \omega$  and the state vector is redefined as  $x = [x_1, \dots, x_n, v, w]$ . Hence, if the original system is a nonautonomous oscillator with periodic forcing, we can also recast it to the autonomous system  $\dot{x} = g(x(t); p)$ .

We can analyze equilibria, their Hopf bifurcations and emanating limit cycles in the possibly recast original system. The original system, together with some information from the limit cycle close to the Hopf bifurcation, is then used to construct and initialize the full continuation system defining the BVP.

#### 3.2. Initialization

As an initial solution for the BVP problem, we could take a precomputed periodic orbit, carry out a signal analysis to find the coefficients of its Fourier expansion, and substitute that into the system. Another way (which we describe here) is to use standard continuation techniques provided by AUTO, following an equilibrium point that undergoes an Andronov–Hopf bifurcation.

In the nonautonomous case with sinusoidal forcing, we can easily find this bifurcation by varying the parameter  $\alpha$  in Eqs. (4) across zero.

The limit cycle that emanates from the Andronov–Hopf bifurcation can then be continued a small distance away from the bifurcation, where it will still have the approximate form

$$x(t) = A \sin \omega t + B \cos \omega t. \tag{5}$$

Here  $\omega$  is the purely imaginary part of the corresponding eigenvalue of the equilibrium, which we can obtain using the period  $T$  that AUTO provides:  $\omega = 2\pi/T$ . Now the Fourier coefficients can be trivially derived, comparing Eqs. (2) and (5):

$$\begin{aligned} a_{0j} &= 0, \\ a_{1j} &= A = \frac{\dot{x}_j(0)}{\omega}, \quad b_{1j} = B = x_j(0), \\ a_{kj} &= 0, \quad b_{kj} = 0, \quad k > 1, \quad j = 1, \dots, n. \end{aligned} \tag{6}$$

### 3.3. The full continuation system

The full continuation system is given by the following equations:

- Nonautonomous differential equations:

$$\begin{aligned} \dot{x} &= Tg(x(t), t; p), & x, g &\in \mathbb{R}^n, \\ p &\in \mathbb{R}^q, & t, T &\in \mathbb{R} \\ \dot{t} &= 1 \end{aligned} \quad (7)$$

Here the possibly recast original system Eq. (1) is rescaled so that a  $T$ -periodic solution of Eq. (1) is a 1-periodic solution for the first equation in Eqs. (7). The nonperiodic equation  $\dot{t} = 1$  is added to make the system look autonomous to continuation software.

- Boundary conditions that define a periodic orbit on the  $t$ -interval  $[0, 1]$ :

$$\begin{aligned} x_j(0) &= x_j(1), & j &= 1, \dots, n \\ t(0) &= 0 \end{aligned} \quad (8)$$

- Integral conditions:

$$\begin{aligned} \int_0^1 x(t)\dot{x}_{\text{old}}(t)dt &= 0 \\ \int_0^1 (2x_j(t)\sin(2\pi kt) - a_{kj})dt &= 0 \\ &\text{for any } a_{kj} \in S_a \\ \int_0^1 (2x_j(t)\cos(2\pi kt) - b_{kj})dt &= 0 \\ &\text{for any } b_{kj} \in S_b \\ \int_0^1 f(S_a, S_b, T, K_{\text{REF}})dt &= 0 \end{aligned} \quad (9)$$

Here the first condition is the standard integral phase condition [Kuznetsov, 2004], where  $x_{\text{old}}(t)$  denotes the previous point on a continuation branch, the second and third conditions compute or fix the Fourier coefficients  $\{a_{kj}\} \in S_a$  and  $\{b_{kj}\} \in S_b$ , and the fourth condition computes or fixes  $K_{\text{REF}}$  given the Fourier coefficients in  $S_a$  and  $S_b$ .

This gives us a system of  $n + 1$  ordinary differential equations,  $n + 1$  boundary conditions and  $n_a + n_b + 2$  integral conditions. Adding a standard pseudo-arclength condition, this needs to be offset by  $n_a + n_b + 3$  continuation parameters.

A basic choice for the parameters is to continue a periodic orbit in one of the system parameters, the period  $T$ , and the  $n_a + n_b + 1$  values in  $S_a \cup S_b \cup \{K_{\text{REF}}\}$ . Note that in this case these last  $n_a + n_b + 1$

values are effectively measured through integral conditions, where explicit test functions such as

$$a_{kj} = \int_0^1 (2x_j(t)\sin(2\pi kt))dt \quad (10)$$

would suffice.

However, the integral conditions become more powerful if we like to keep something fixed: for instance, by fixing  $K_{\text{REF}}$  and freeing up one more system parameter we can continue iso-amplitude curves. Moreover, in existing continuation software it is easier and arguably more elegant, if a little more computationally expensive, to stick to one full system with the same integral conditions, than to switch between test functions and equivalent integral conditions.

## 4. Case Study 1: An Autonomous Electronic Oscillator

Increasing demand, in modern communication systems, of high-performance low-power radio-frequency circuits is driving the development of accurate simulation tools at the design stage. The simulation is particularly critical in the case of analog circuits (free-running or voltage-controlled oscillators), because of time consumption, and in the case of mixed-signal analog-digital circuits (frequency synthesizers or phase-locked loops), due to the modeling difficulties. In all cases, the analysis is usually nontrivial and performance verification requires extensive simulation. This is even more important when the main goal is to find how changes in circuit parameters (for example, the amplitude or frequency of an input generator, or a linear-element value) affect circuit performance.

The main research lines in this field are twofold. Firstly there is the development of algorithms that speed up and make circuit simulations more reliable [Brambilla & Storti-Gajani, 2005; Brambilla et al., 2008]. Secondly there is the study of methods that, starting from a simplified version of the designed circuit, aim to provide design criteria avoiding brute-force simulations of complex integrated circuits. Most of these methods are based on well-established theories such as harmonic balance, Volterra series (for weakly nonlinear oscillators), and bifurcation analysis of nonlinear dynamical systems. Harmonic balance is used for an accurate determination of nonlinear-circuit operation bands [Rizzoli & Neri, 1998; Suárez et al., 1998] and, associated with other methods, for calculating the periodic response of nonlinear dynamical

circuits [Buonomo & Lo Schiavo, 2003]. Volterra series are used for the analysis of nearly sinusoidal nonlinear oscillators [Hu *et al.*, 1989; Huang & Chu, 1994]. Bifurcation analysis is used to optimize the design of oscillators [Maggio *et al.*, 1999] and the locking range of injection-locked frequency dividers [Ghahramani *et al.*, 2007]. A combination of harmonic-balance simulators and bifurcation control is exploited to obtain bifurcation control in microwave circuits, thus presetting the operation bands of complex circuits, such as synchronized and voltage-controlled oscillators and frequency dividers [Collado & Suárez, 2005]. This combination is also used to obtain robust and efficient oscillator analysis techniques (see, for example [Bonani & Gilli, 1999; Genesio *et al.*, 1993; Gourary *et al.*, 2000] and references therein).

Our case study is the Colpitts oscillator, shown in Fig. 1, whose dynamics were analyzed in detail in [Maggio *et al.*, 1999; De Feo *et al.*, 2000; De Feo & Maggio, 2003]. The following simplified model, that we can easily substitute into the system defined in Sec. 3, adequately describes this oscillator:

$$\begin{cases} \dot{x} = \frac{G}{Q(1-\gamma)}[-\alpha_F(e^{-y}-1) + z] \\ \dot{y} = \frac{G}{Q\gamma}[(1-\alpha_F)(e^{-y}-1) + z] \\ \dot{z} = -\frac{Q\gamma(1-\gamma)}{G}(x+y) - \frac{1}{Q}z \end{cases}, \quad (11)$$

where the system parameters are initialized as follows:

$$Q = \frac{\omega_0 L}{R} = 0.8; \quad \gamma = \frac{C_2}{C_1 + C_2};$$

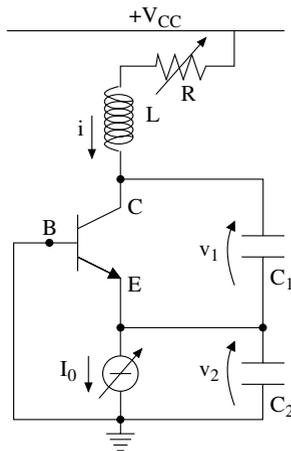


Fig. 1. The considered Colpitts oscillator.  $V_{CC} = 3$  V,  $C_1 = C_2 = 1$   $\mu$ F,  $L = 10$  mH.

$$G = \frac{I_0 L}{V_T R(C_1 + C_2)} = 2;$$

$$C_1 = C_2 = 10^{-6}[F]; \quad L = 10^{-3}[H]. \quad (12)$$

Moreover, we assume  $\alpha_F = 1$ , that is, we assume the CB short-circuit forward current gain of the BJT transistor to be ideal. The variable  $z$  denotes the inductor current normalized with respect to  $I_0$ . The variables  $x$  and  $y$  are the voltages across  $C_1$  and  $C_2$  normalized with respect to  $V_T = 25.9$  mV (that is, the thermal voltage at room temperature). Time is normalized with respect to  $T_0 = \sqrt{LC_1 C_2 / (C_1 + C_2)}$ .

Continuing the equilibrium at 0 as the parameter  $G$  goes downwards from 2 we find a Hopf bifurcation at  $G = 1$ . For  $G$  slightly greater than 1 there then exists a limit cycle for which the Fourier coefficients  $a_{12}$ ,  $b_{12}$  and  $K_{\text{REF}}$  are given by [see Eqs. (6)]

$$a_{12} = \frac{\dot{y}(0)}{\frac{2\pi}{T}} = \frac{Gz(0)}{\frac{2\pi}{T}},$$

$$b_{12} = y(0), \quad (13)$$

$$K_{\text{REF}} = \sqrt{a_{12}^2 + b_{12}^2}.$$

We focus on the state variable  $y$  since it corresponds to the voltage used as output of the oscillator. Moreover, we focus on its first harmonic component since we want the generated oscillation to be nearly sinusoidal. Thus, we need to verify that higher order harmonics have small amplitudes when compared with the first harmonic component.

We then follow the periodic orbits as a boundary value problem in  $(G, T, a_{12}, b_{12}, K_{\text{REF}})$  with user-defined labels for the following values of  $K_{\text{REF}}$ : 1, 4, 7, 10, 13, 16, 19, 22, 25. Using these values as starting points we can find iso-harmonic curves in  $(G, \gamma)$ -parameter plane for fixed values of  $K_{\text{REF}}$  by continuing in  $(Q, G, T, a_{12}, b_{12})$ .

We carried out the continuation in the parameter space  $(Q, G, T, a_{12}, b_{12})$ , but show the results in Fig. 2 on the plane  $(R, I_0)$ , according to the definitions of  $Q$  and  $G$ , to simplify the interpretation of the results from a circuit point of view. We excluded the parameter space region characterized by the presence of complex dynamics [Maggio *et al.*, 1999; De Feo *et al.*, 2000] from our analysis and focused on the region characterized by the presence of a stable “simple” harmonic cycle.

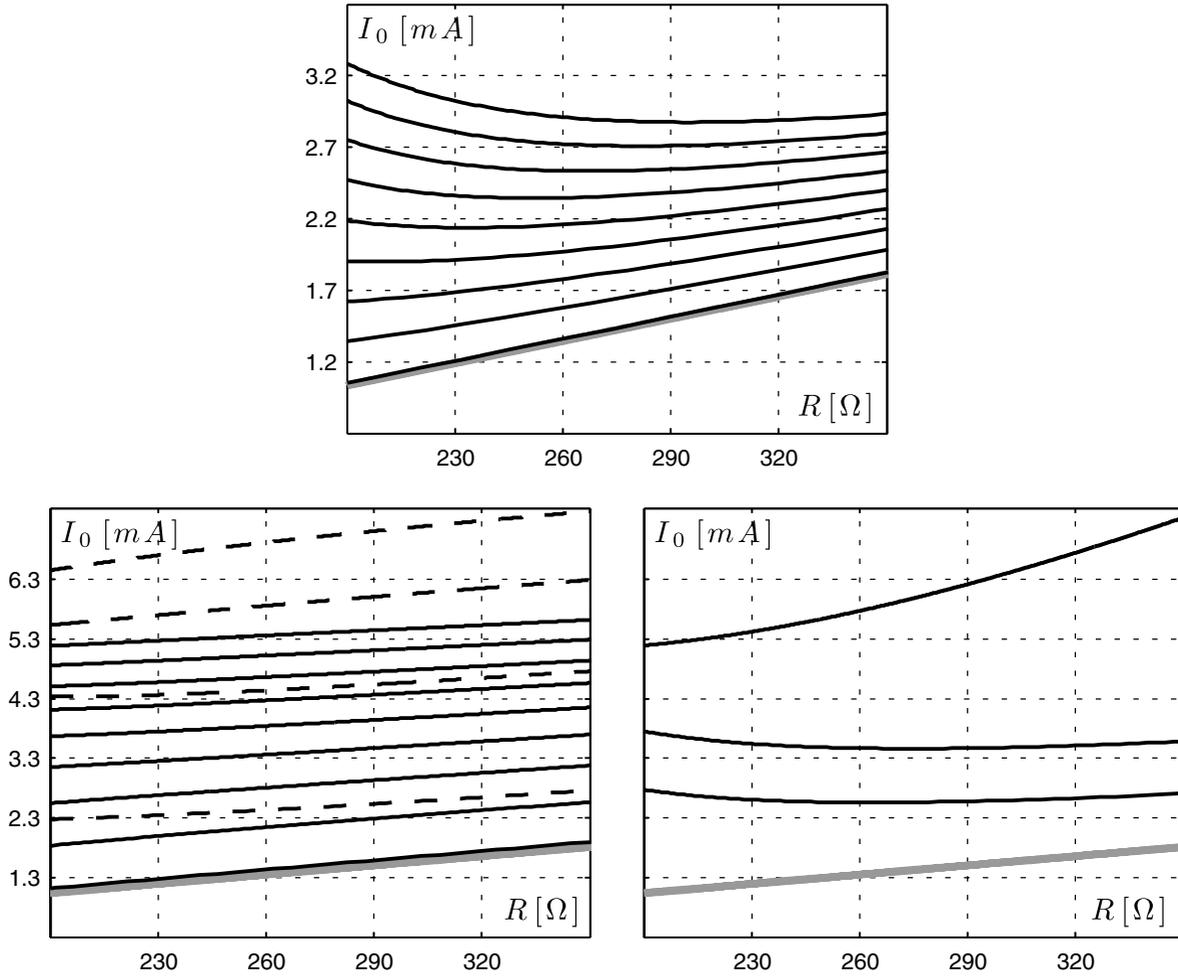


Fig. 2. Charts for the Colpitts oscillator on the parameter plane  $(R, I_0)$ . The grey line is the Hopf bifurcation curve in all panels. Iso-period curves (upper panel), iso- $A_1$  (solid) and iso- $A_2$  (dashed) curves (lower-left panel), and iso-ratio  $A_1/A_2$  (lower-right panel).

In the upper panel, along the black dashed curves the period (normalized with respect to  $T_0$ ) is constant: 6.3 for the lower curve, and 8.7 for the upper curve, with a step of 0.3. In the lower-left panel, along the black solid curves the amplitude  $A_1$  (normalized with respect to  $V_T$ ) of the first harmonic component of  $y$  is constant (1 for the lower curve, 25 for the upper curve, step of 3), whereas along the black dashed curves the amplitude  $A_2$  (normalized with respect to  $V_T$ ) of the second harmonic component of  $y$  is constant (1, 5, 10 and 15 from the lower curve to the upper one).

In the left-right panel, we follow the periodic orbits as a boundary value problem in  $(G, T, a_{12}, b_{12}, a_{22}, b_{22}, K_{REF})$ . Along the black solid curves the ratio  $A_1/A_2 (= K_{REF})$  is constant: 3 for the upper curve, 5 for the lower curve, with a step of 1. The interpretation of these results is

straightforward: for instance, if we want to fix both circuit parameters to have an oscillation frequency  $f = T_0/6.3$ , and ensure low values of  $R$  to increase the quality factor  $Q$ , we can just properly adjust  $I_0$  along the lower curve in the upper panel of Fig. 2. The results shown in the lower panels provide useful information if a periodic signal whose first harmonic component is predominant with respect to the second one is of interest.

Of course, we can obtain other iso-curves besides those shown in Fig. 2. For example, we may choose values of  $I_0$  and  $R$  ensuring a desired ratio  $A_p/A_q$  for given  $p$  and  $q$ .

Figure 3 shows the obtained iso-harmonic (with constant  $A_1$ ) curves in the parameter subspace  $(R, I_0, T)$ . Of course, the projection of the curves on the plane  $(R, I_0)$  gives the black solid curves displayed in the lower-left panel of Fig. 2.

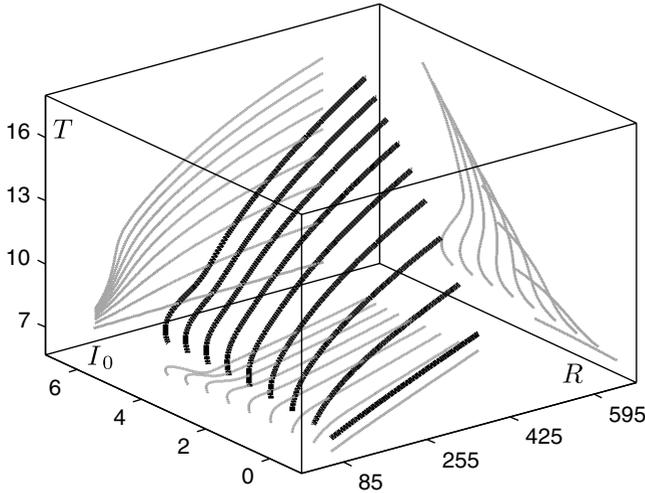


Fig. 3. Iso-harmonic curves in the parameter subspace  $(R, I_0, T)$ .

### 5. Case Study 2: A Nonautonomous Mechanical Oscillator

The nonlinear damped oscillator is a model widely used to represent shock absorbers [Lang *et al.*, 2006] and is given by the following system (with dimensionless variables and parameters):

$$\begin{aligned} \dot{x} &= \frac{A}{m} \cos(\omega t) \\ &\quad - \frac{1}{m}(c_1 x + c_2 x^2 + c_3 x^3 + ky) \\ \dot{y} &= x \end{aligned} \tag{14}$$

where  $m = 240$ ,  $c_1 = 296$ ,  $c_2 = 3000$ ,  $c_3 = 800$ , and  $k = 240(4\pi)^2$ . The reference angular frequency of the system is  $\omega_0 = \sqrt{k/m} = 4\pi$ .

We can easily recast the nonautonomous system as an autonomous system by adding the auxiliary oscillator (4) to Eqs. (14). For  $\alpha < 0$ , the resulting system has a stable equilibrium at the origin, which undergoes a Hopf bifurcation for  $\alpha = 0$ . For small positive  $\alpha$  there then exists a limit cycle for which the Fourier coefficients  $a_{12}$ ,  $b_{12}$ , and  $K_{\text{REF}}$  are approximately given by, and determined numerically as [see Eqs. (6)]

$$\begin{aligned} a_{12} &= \frac{\dot{y}(0)}{2\pi} = \frac{x(0)}{T}, \\ b_{12} &= y(0), \end{aligned} \tag{15}$$

$$K_{\text{REF}} = \sqrt{a_{12}^2 + b_{12}^2}.$$

We focus on the state variable  $y$  since it represents the position of the mechanical oscillator.

Once the initialization is completed, we work with the full continuation system and continue the limit cycle with respect to  $(\alpha, T, a_{11}, b_{11}, K_{\text{REF}})$  until we reach  $\alpha = 1$ , corresponding to the correct sinusoidal forcing.

In this case, we monitor the amplitude of the first harmonic with respect to one of the parameters  $\omega$  and  $A$  (properly normalized), besides  $(T, a_{11}, b_{11}, K_{\text{REF}})$ , to provide evidence for the

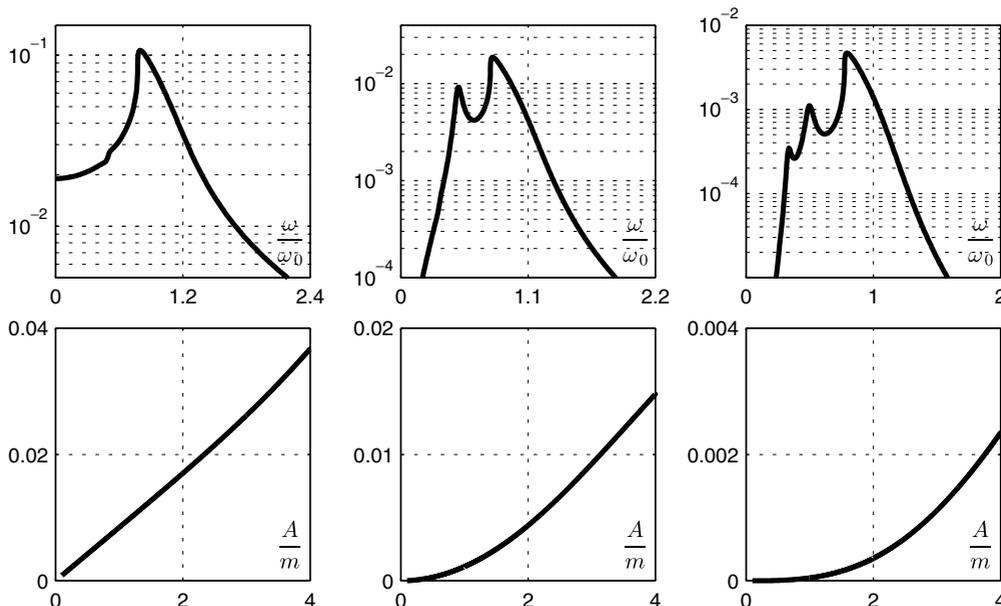


Fig. 4. Amplitudes of the first (first column), second (second column) and third (third column) harmonics with respect to  $\omega/\omega_0$  (upper panels) and to  $A/m$  (lower panels).

presence of a “jump phenomenon”. The “jump phenomenon” is a characteristic feature of many nonlinear oscillators, where the response amplitude changes suddenly at some critical value of the excitation frequency [Schmidt & Tondl, 1986]. Many widely used methods such as the Harmonic Balance (HBM) and Nonlinear Output Frequency Response Function (NOFRF) [Peng *et al.*, 2008] barely capture this phenomenon. Figure 4 shows the amplitudes of the first three harmonics as functions of the normalized frequency  $\omega/\omega_0$  (upper panels) and of the normalized amplitude  $A/m$  (lower panels). The results perfectly match the reference diagrams reported in [Peng *et al.*, 2008].

We remark that the proposed continuation method does not require any approximation, thus providing excellent accuracy. Moreover, it is based on tools (such as AUTO) that are reliable and widely tested.

Finally, by performing continuations similar to those shown for the Colpitts oscillator, we can also obtain simulations pertaining to the design of mechanical oscillators, that is, we obtain the values of parameters that ensure a desired behavior.

## 6. Conclusions

We proposed a technique, based on numerical continuation, that enables harmonic analysis of a given nonlinear oscillator without resorting to any approximation. Moreover, a designer can choose some oscillator parameters to obtain a desired behavior, by analyzing some of the curves that are obtained by this technique. More realistically, since the model is an approximation of the real system, the proposed method provides at least reference values of the bifurcation parameters. More accurate simulations focused on restricted portions of the parameter space can then refine these values.

The main advantage of this technique is that it enables the analysis of even relatively complex oscillators (both autonomous and forced) by using software tools that are reliable and optimized. This makes the development of *ad hoc* software for this kind of analysis unnecessary and ensures an excellent accuracy of the results.

The main limit of this technique is that it requires a thorough knowledge of continuation methods and/or software packages for numerical continuation. Moreover, for oscillators forced by nonsinusoidal periodic signals, it can be nontrivial to define the auxiliary equations needed to make

the system autonomous. In these cases, it may be more convenient to use the results of a signal analysis carried out on a precomputed periodic orbit. The coefficients of the thus obtained Fourier expansion can be substituted into the system, which can then be used as an initial solution for the BVP problem.

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